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The dihedral coverings of the projective plane

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Introduction

Let r be an odd integer greater than 2 and let

$$D_{2r} = \langle \sigma, \tau \mid \sigma^2 = \tau^r = (\sigma\tau)^2 = 1 \rangle.$$

Then the following fact is well-known(see [3] or [5]). If g_1 and g_2 are homogeneous polynomials of three variables with $2 \deg g_1 = r \deg g_2$, then there exists a D_{2r}-covering of $\mathbf{P}^2$ ramifying only along the curve C defined by $g_1^2 - g_2^r = 0$, where a D_{2r}-covering is a (branched) Galois covering with the Galois group isomorphic to D_{2r}. Moreover, if (g_1) crosses (g_2) normally at $\deg g_1 \deg g_2$ points, then C has $(2, r)$ cusps there. In this note, we show that if there exists a D_{2r}-covering of $\mathbf{P}^2$ ramifying only along an irreducible reduced curve $C = (f)$, then there exist homogeneous polynomials h, g_1 and g_2 of three variables, satisfying

$$fh^2 = g_1^2 - g_2^r.$$

Conversely, we also show that if there exist homogeneous polynomials f, h, g_1 and g_2 of three variables satisfying the above equation, if (f) contains no irreducible components of (h) and if (g_1) crosses (g_2) normally at at least one point, then there exists a D_{2r}-covering of $\mathbf{P}^2$ ramifying only along (f). As an application, we give an example of a D_{10}-covering ramifying along a sextic with four $(2, 5)$ cusps.

1 A versal dihedral covering

We define the action of the dihedral group D_{2r} on $\mathbf{P}^1$ by

$$\sigma : \xi \mapsto \xi^{-1} \quad \text{and} \quad \tau : \xi \mapsto \rho_r \xi,$$

where $\rho_r = \exp(2\pi\sqrt{-1}/r)$ and ξ is a non-homogeneous coordinate of $\mathbf{P}^1$. Then the holomorphic map

$$\varpi : \mathbf{P}^1 \ni \xi \mapsto \xi^r + \xi^{-r} \in \mathbf{P}^1$$

is a D_{2r}-covering. This covering plays a key role in this note. Let $\nu : Y \to \mathbf{P}^1$ be a dominant rational map from a projective variety Y and let Y_0 be the complement of the set of the indeterminacy of ν. Then we have the following(see §4 in [2]).

Proposition 1. *If the fiber product $Y_0 \times_{\mathbf{P}^1} \mathbf{P}^1$ of the restriction $\nu_{|Y_0}$ of ν to Y_0 and ϖ, is irreducible, then there exists a D_{2r}-covering $\pi : X \to Y$ and a D_{2r}-equivariant rational map $\mu : X \to \mathbf{P}^1$ with $\varpi \circ \mu = \nu \circ \pi$.*

$$\begin{array}{ccccc} X & \xrightarrow{\mu} & \mathbf{P}^1 & \ni & \xi \\ \pi \downarrow & & \downarrow \varpi & & \downarrow \\ Y & \xrightarrow{\nu} & \mathbf{P}^1 & \ni & \xi^r + \xi^{-r} \end{array}$$

Conversely, any D_{2r}-covering of a projective variety can be constructed in this way provided that r is odd.

Theorem 2. *Let r be an odd integer. For any D_{2r}-covering $\pi : X \to Y$ of a projective variety Y, there exist a D_{2r}-equivariant rational map $\mu : X \to \mathbf{P}^1$ and a dominant rational map $\nu : Y \to \mathbf{P}^1$ with $\varpi \circ \mu = \nu \circ \pi$.*

Proof. Let f_0 be a rational function on X and let

$$f_1 = \sum_{i=0}^{r-1} \frac{(\tau^i)^* f_0}{\rho_r^{is}}, \qquad f = \frac{f_1}{\sigma^* f_1}$$

where $\rho_r = \exp(2\pi\sqrt{-1}/r)$ and $s = (r+1)/2$. Then $\tau^* f_1 = \rho_r^s f_1$ and $\sigma^* f = f^{-1}$. Hence $\tau^* f = \rho_r f$, because $\tau^*\sigma^* f_1 = \sigma^*(\tau^{-1})^* f_1 = \rho_r^{-s}\sigma^* f_1$ and $\rho_r^{2s} = \rho_r$. Therefore, the rational function f defines a D_{2r}-equivariant rational map $\mu : X \to \mathbf{P}^1$. Moreover, f is not constant for a suitable f_0. Hence the rational map $\nu : Y \to \mathbf{P}^1$ defined by $f^r + f^{-r}$, is dominant and $\varpi \circ \mu = \nu \circ \pi$. □

Remark. It is well-known that there exist G-coverings $\mathbf{P}^1 \to \mathbf{P}^1$ also for the groups G isomorphic to D_{2r} with even r, A_4, S_4 and A_5. However, the above theorem does not hold for these coverings(see [1]).

2 Dihedral coverings ramifying along irreducible curves

We keep the notations in the previous section.

Proposition 3. *Let $C = (f)$ be an irreducible reduced curve on $\mathbf{P}^2$ defined by a homogeneous polynomial f. If there exists a D_{2r}-covering $\pi : Y \to \mathbf{P}^2$ of $\mathbf{P}^2$ ramifying only along C, then there exist homogeneous polynomials h, g_1 and g_2 satisfying $fh^2 = g_1^2 - g_2^r$, and π is induced from ϖ and the rational map $\mathbf{P}^2 \to \mathbf{P}^1$ defined by $\pm(4g_1^2/g_2^r - 2)$.*

Proof. The ramification index of π along C is equal to 2, because the double covering $Y/\langle\tau\rangle \to \mathbf{P}^2$ ramifies along C and any element in D_{2r} not contained in $\langle\tau\rangle$, has order 2. On the other hand, ϖ ramifies at 2, -2 and ∞ with the ramification index 2, 2 and

r, respectively. Hence there exists a dominant rational map $\nu : \mathbf{P}^2 \to \mathbf{P}^1$ such that $\nu(C) = 2$ or -2, by Theorem 2. Let $\psi : \mathbf{P}^1 \to \mathbf{P}^1$ be the biholomorphic map defined by $(\xi + 2)/4$ or $(-\xi + 2)/4$, accordingly as $\nu(C) = 2$ or -2. Then $(\psi \circ \nu)(C) = 1$. There exist homogeneous polynomials $\tilde{g}_1$ and $\tilde{g}_2$ such that $\deg \tilde{g}_1 = \deg \tilde{g}_2$ and that $\psi \circ \nu$ is defined by $\tilde{g}_1/\tilde{g}_2$. Since ϖ ramifies along $\psi^{-1}(\infty) = \infty$ and $\psi^{-1}(0) = \mp 2$ with the ramification index r and 2, respectively, and π does not ramify along $(\psi \circ \nu)^{-1}(\infty) = (\tilde{g}_2)$ and $(\psi \circ \nu)^{-1}(0) = (\tilde{g}_1)$, there exist homogeneous polynomials g_1 and g_2 with $\tilde{g}_1 = g_1^2$ and $\tilde{g}_2 = g_2^r$. Since $(\psi \circ \nu)^{-1}(1) = (g_1^2 - g_2^r) \supset C = (f)$ and π ramifies only along (f), there exists a homogeneous polynomial h with $fh^2 = g_1^2 - g_2^r$. □

Remark. For any homogeneous polynomial f of even degree there exist homogeneous polynomials h, g_1 and g_2 satisfying $fh^2 = g_1^2 - g_2^r$. For example, $h = \binom{r}{1} l^{r-1} + \binom{r}{3} l^{r-3} f + \cdots + f^{(r-1)/2}$, $g_1 = l^r + \binom{r}{2} l^{r-2} f + \cdots + \binom{r}{r-1} l f^{(r-1)/2}$ and $g_2 = l^2 - f$ satisfy the equality $g_2^r = g_1^2 - fh^2$ for any homogeneous polynomial l with $\deg l = \deg f/2$, because $(l \pm \sqrt{f})^r = g_1 \pm \sqrt{f} h$. However, then

$$4\frac{g_1^2}{g_2^r} - 2 = \left(2\frac{l^2 + f}{g_2}\right)^r + c_2 \left(2\frac{l^2 + f}{g_2}\right)^{r-2} + \cdots + c_{r-1} \left(2\frac{l^2 + f}{g_2}\right),$$

where c_i are the integers determined by the equation

$$\xi^r + \xi^{-r} = (\xi + \xi^{-1})^r + c_2(\xi + \xi^{-1})^{r-2} + \cdots + c_{r-1}(\xi + \xi^{-1}).$$

Hence the rational map $\nu : \mathbf{P}^2 \to \mathbf{P}^1$ defined by $4g_1^2/g_2^r - 2$, is equal to the composite $\varpi' \circ \nu'$ of the rational map $\nu' : \mathbf{P}^2 \to \mathbf{P}^1/\langle\sigma\rangle \simeq \mathbf{P}^1$ defined by $2(l^2 + f)/g_2$ and the holomorphic map $\varpi' : \mathbf{P}^1/\langle\sigma\rangle \to \mathbf{P}^1$ induced from ϖ. Therefore, the fiber product $Z_0 \times_{\mathbf{P}^1} \mathbf{P}^1$ of the restriction of ν to Z_0 and ϖ, is reducible, where Z_0 is the complement of the set of points of indeterminacy of ν.

Proposition 4. *Let $C = (f)$ be a reduced curve on $\mathbf{P}^2$ defined by a homogeneous polynomial f. Assume that there exist homogeneous polynomials g_1, g_2 and h satisfying $fh^2 = g_1^2 - g_2^r$. If C contains no irreducible components of the zero divisor (h) of h and (g_1) crosses (g_2) normally at at least one point, then there exists a D_{2r}-covering of $\mathbf{P}^2$ ramifying only along C.*

Proof. Let $\nu : \mathbf{P}^2 \to \mathbf{P}^1$ be the rational map defined by $4g_1^2/g_2^r - 2$. Since ν is dominant, there exists a D_{2r}-covering of $\mathbf{P}^2$ ramifying only along C, if and only if the fiber product $Z_0 \times_{\mathbf{P}^1} \mathbf{P}^1$ of the restriction of ν to Z_0 and ϖ, is irreducible, where Z_0 is the complement of the set of points of indeterminacy of ν.

Let (g_1) cross (g_2) normally at p, let W be a small neighborhood of p and let l be a linear equation with $(l) \cap W = \emptyset$. Then $((g_1/l^{\deg(g_1)})_{|W}, (g_2/l^{\deg(g_2)})_{|W})$ is a local coordinate system of W and there exists a D_{2r}-covering $\pi : U \to W$ which is expressed as $(u, v) \mapsto ((u^r + v^r)/2, uv)$ by a local coordinate system (u, v) of U, where U is an open neighborhood of the origin in $\mathbf{C}^2$. Let $\mu : U \to \mathbf{P}^1$ be the meromorphic map defined by

u/v. Then μ is D_{2r}-equivariant and $\varpi \circ \mu = \nu_{|W} \circ \pi$.

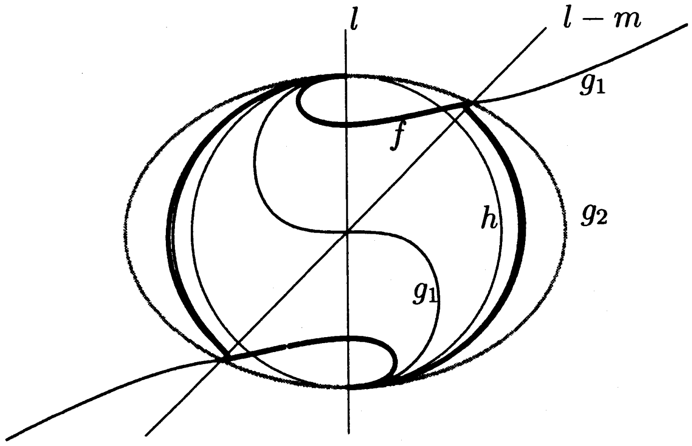
$$\begin{array}{ccccccc} U \ni & (u,v) & \xrightarrow{\mu} & \dfrac{u}{v} & = & \xi & \in \mathbf{P}^1 \\ & \pi \downarrow & & & & \downarrow \varpi & \\ W \ni & (\dfrac{u^r+v^r}{2}, uv) & = (x,y) & \xrightarrow{\nu_{|W}} & 4\dfrac{x^2}{y^r} - 2 & = & \xi^r + \xi^{-r} & \in \mathbf{P}^1 \end{array}$$

Hence $(W \setminus \{p\}) \times_{\mathbf{P}^1} \mathbf{P}^1$ is irreducible. Therefore, also is $Z_0 \times_{\mathbf{P}^1} \mathbf{P}^1$.

The D_{2r}-covering of $\mathbf{P}^2$ induced from ν and ϖ, ramifies only along C, because $\nu^{-1}(-2) = 2(g_1)$, $\nu^{-1}(\infty) = r(g_2)$ and $\nu^{-1}(2) = (f) + 2(h)$. □

3 An example

Let l, m and h be homogeneous polynomials of degree 1, 1 and 2, respectively, with $(l) \cap (m) \cap (h) = \emptyset$. Let $g_1 = l^5 - 5l^3h + 6mh^2$ and let $g_2 = l^2 - 2h$. Then $g_1^2 - g_2^5 = h^2 f$, where $f = -15l^6 + 12l^5m + 80l^4h - 60l^3mh - 80l^2h^2 + 36m^2h^2 + 32h^3$. Assume that $(l-m)$ crosses (g_2) normally at two points. Then (g_1) also does, because $g_1 = lg_2(l^2 - 3h) - 6(l-m)h^2$ and $(l-m) \cap (g_2) \cap (h) = \emptyset$. Hence there exists a D_{10}-covering of $\mathbf{P}^2$ ramifying along the sextic curve (f). We easily see that if (l) crosses (h) normally, then (f) has two $(2,5)$ cusps at $(l) \cdot (h)$ as well as at $(l-m) \cdot (g_2)$.



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